# Crossing Regular Cycle Drawings

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# Crossing Regular Cycle Drawings

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Drawings $D(G)$ of a graph $G$ are realizations of $G$ in the plane where the vertices of $G$ are mapped into different points of the plane (also called vertices of $D(G)$) and the edges are mapped into simple curves (also called edges of $D(G)$) which connect corresponding points in such a way that two curves have at most one point in common, either an endpoint or a crossing.

Here the special corssing $r$-regular drawings $D(C_n)$ of the cycle graph $C_n$ are discussed where every of its $n$ edges contains the same number $r$ of crossings. Planar $D(C_n)$ which have the minimum number 0 of crossings are examples for $r = 0$. In drawings $D(C_n)$ with the maximum number $CR(C_n) = \frac{n\,(n-3)}{2}$ of crossings $(n \geq 5)$ every edge contains $r = n - 3$ crossings (see [1]). What about crossing $r$-regular $D(C_n)$ for values $r$ with $0 < r < \frac{n\,(n-3)}{2}$? Since $nr$ counts twice the number of all crossings in $D(C_n)$ either $n$ or $r$ has to be even for the existence of a crossing $r$-regular $D(C_n)$. Thus the following question arises.

**Problem.** Do crossing $r$-regular drawings $D(C_n)$ exist for $n \geq 5$
if $n \equiv 0 \pmod 2$ for $0 \leq r \leq n-3$, and
if $n \equiv 1 \pmod 2$ for $0 \leq r \leq n-3$, $r \equiv 1 \pmod 2$?

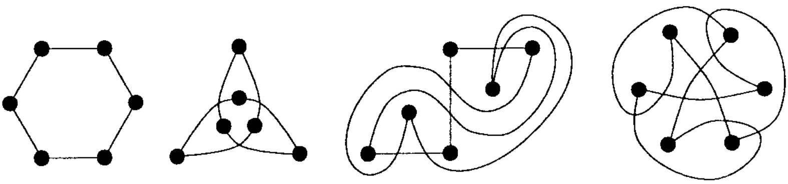

*Figure 1*

For example, Figure 1 illustrates the existence in all cases of $C_6$. In the following some general classes of crossing $r$-regular $D(C_n)$ are presented.

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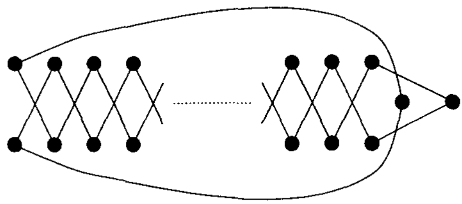

*Figure 2*

For $r = 1$ a solution is given by Figure 2.

In general it suffices to prove the existence for $r + 3 \leq n \leq 2r + 5$ since for $j \geq 6$ a crossing $r$-regular $D(C_{r+3})$ and a crossing $r$-regular $D(C_{r+j-3})$ can be combined as in Figure 3 where two vertices, one vertex of each drawing, are chosen close to one another and then substituted by those vertices with the dotted edges.

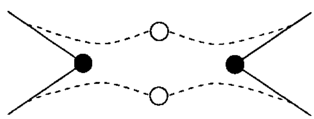

*Figure 3*

**Figure 4**: For $r \equiv 0 \pmod 2$ and $n$ with $\left(n, \frac{r}{2} + 1\right) = 1$ the vertices of a convex $n$-gon are used together with its diagonals of distance $\frac{r}{2} + 1$ (see Figure 4 for $C_8$, $r = 4$).

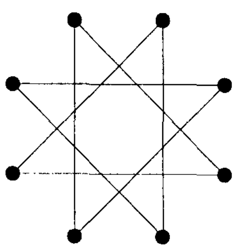

*Figure 4*

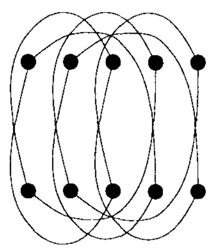

*Figure 5*

**Figure 5**: For $r \equiv 1 \pmod 2$, $n \equiv 0 \pmod 2$, $n \geq 2r$, and $\left(\frac{n}{2}, r - 1\right) = 1$ the vertices are chosen as $\frac{n}{2}$ opposite pairs of points on two parallels. Every vertex is connected to two consecutive vertices on the other parallel which are $\frac{r-1}{2}$ and $\frac{r+1}{2}$ positions distant as in Figure 5 for $C_{10}$, $r = 5$.

**Figure 6**: If $D\left(C_{\frac{n}{t}}\right)$ exists crossing $\left(\frac{r}{t}\right)$-regular with $(r, n) \geq t$ then $D(C_n)$ exists crossing $r$-regular. Use $t$ nearly parallel copies of $D\left(C_{\frac{n}{t}}\right)$ and $t$ corresponding vertices in an appropriate way (see Figure 6 for $C_{15}$, $r = 6$, $t = 3$).

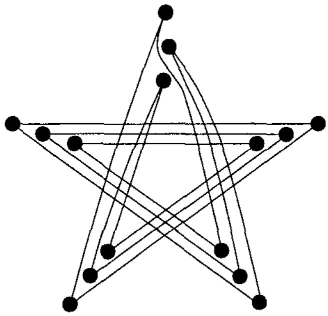

*Figure 6*

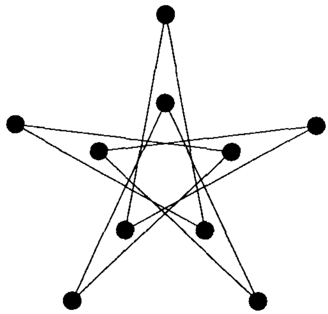

*Figure 7*

**Figure 7**: If $D\left(C_{\frac{n}{2}}\right)$ exists crossing $\left(\frac{r-1}{2}\right)$-regular for $r \equiv 1 \pmod 2$, $n \equiv 0 \pmod 2$ then $D(C_n)$ exists crossing $r$-regular. Use the construction of Figure 6 for $t = 2$ and let parallel edges intersect one another (see Figure 7 for $C_{10}$, $r = 5$).

**Figure 8**: For $n \equiv 0 \pmod 2$, $n \geq 2r + 4$, $\left(\frac{n}{2}, r + 1\right) = 1$ on each of two concentric circles $\frac{n}{2}$ vertices are drawn. Then every vertex of the outer circle is connected to the points to right and to left of $r$ consecutive vertices on the inner circle (see Figure 8 for $C_{10}$, $r = 2$ and $r = 3$).

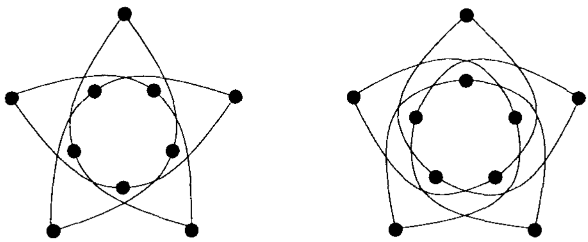

*Figure 8*

**Figure 9**: For $n \equiv 0 \pmod 4$, $n \geq 4r$, $r \geq 2$, $\left(\frac{n}{4}, r - 1\right) = 1$ two crossing $(r - 2)$-regular copies of $D\left(C_{\frac{n}{2}}\right)$ of the type in Figure 8 can be used if corresponding rays of the star-like drawings are intersected and then two edges of two corresponding vertices interchanged (see Figure 9 for $C_{12}$, $r = 3$).

**Figure 10**: For $n \equiv 0 \pmod 2$, $r \equiv 0 \pmod 4$, $n \geq r+4$, $\left(\frac{n}{2}, \frac{r}{2}+1\right) = 1$ two concentric circles with $\frac{n}{2}$ vertices each are used. Every vertex of the outer circle is connected to two vertices of the inner circle going into the inner circle between the two opposite neighbor vertices and then passing $\frac{r}{4}$ vertices to the right and to the left (see Figure 10 for $C_{14}$, $r = 8$).

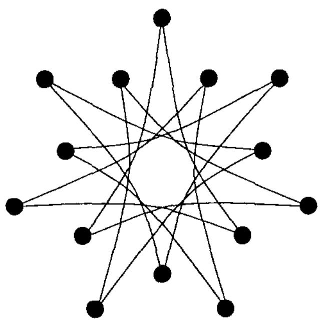

*Figure 10*

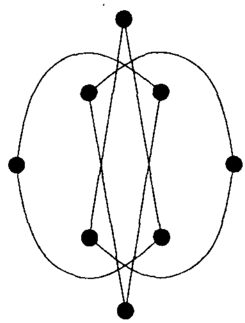

*Figure 11*

Special crossing $r$-regular $D(C_n)$ are presented in Figure 11 for $C_8$, $r = 3$, in Figure 12 for $C_9$, $r = 4$, in Figure 13 for $C_{12}$, $r = 5$, in Figure 14 for $C_{16}$, $r = 7$, in Figure 15 for $C_{15}$, $r = 8$, and in Figure 16 for $C_{16}$, $r = 9$.

In Table 1 the numbers refer to the drawings in the corresponding Figures and 0 means a $D(C_n)$ with the maximum number $CR(C_n)$ of crossings. It follows that the only missing cases for $r \leq 9$ are $D(C_{10})$ and $D(C_{14})$ both for $r = 6$.

As another class of graphs paths $P_{n+1}$ are crossing $r$-regular if $C_n$ is crossing $r$-regular since two vertices may be separated into two vertices. Further problems on crossing $r$-regular *drawings of graphs are discussed in [2]. It may be concluded with the question to the reader whether he can find a drawing of a 10-gon such that every edge is intersected exactly* 6 times.

| | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | – | 8 | – | ... | | | | | | | | | | | | | | | | | |
| 2 | 0 | 1 | 4 | 8 | 4 | ... | | | | | | | | | | | | | | | |
| 3 | | 0 | – | 11 | – | 5 | – | ... | | | | | | | | | | | | | |
| 4 | | | 0 | 4 | 12 | 4 | 4 | 8 | 4 | ... | | | | | | | | | | | |
| 5 | | | | 0 | – | 5 | – | 13 | – | 5 | – | ... | | | | | | | | | |
| 6 | | | | | 0 | ? | 4 | 6 | 4 | ? | 4 | 8 | 4 | ... | | | | | | | |
| 7 | | | | | | 0 | – | 7 | – | 5 | – | 14 | – | 8 | – | ... | | | | | |
| 8 | | | | | | | 0 | 4 | 4 | 4 | 15 | 4 | 4 | 4 | 4 | 8 | 4 | ... | | | |
| 9 | | | | | | | | 0 | – | 16 | – | 7 | – | 5 | – | 7 | – | 5 | – | ... | |
| 10 | | | | | | | | | 0 | ? | ? | 6 | 4 | ? | 4 | 6 | ? | ? | 4 | 8 | 4 |

*Table 1*

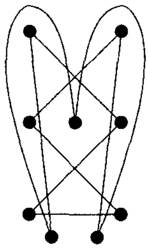

*Figure 12*

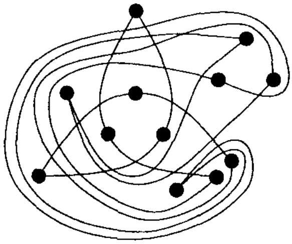

*Figure 13*

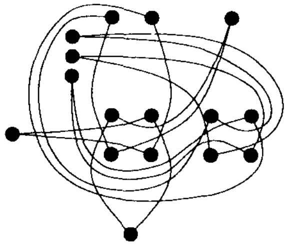

*Figure 14*

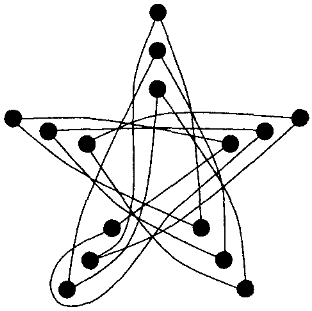

*Figure 15*

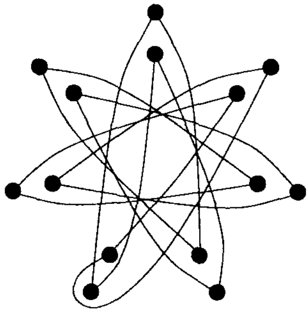

*Figure 16*

## References

[1] H. Harborth: Drawings of the cycle graph. Congressus Numerantium **66** (1988), 15–22.
[2] H. Harborth and I. Mengersen: Crossing regular drawings of graphs. *ZAMM (submitted).*